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ASSESSMENT OF COMPUTATIONAL MODELS FOR MULTILAYERED COMPOSITE CYLINDERS

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Abstract—A study is made of the effects of variation in the lamination and geometric parameters of multilayered composite cylinders on the accuracy of the static and vibrational responses predicted by eight modeling approaches, based on two-dimensional shear-deformation shell theories. The standard of comparison is taken to be the *exact* three-dimensional elasticity solutions, and the quantities compared include both the gross response characteristics (e.g. vibration frequencies, strain energy components, average through-the-thickness displacements and rotations); and detailed, through-the-thickness, distributions of displacements, stresses and strain energy densities. Based on the numerical studies conducted, a predictor-corrector approach, used in conjunction with the first-order shear-deformation theory (with five displacement parameters in the predictor phase), appears to be the most effective among the eight modeling approaches considered. For multilayered orthotropic cylinders the response quantities obtained by the predictor-corrector approach are shown to be in close agreement with the exact three-dimensional elasticity solutions for a wide range of lamination and geometric parameters. The potential of this approach for predicting the response of multilayered shells with complicated geometry is also discussed.

1. INTRODUCTION

Since the first reported investigation of anisotropic shells of revolution (Shtayerman, 1924), considerable progress has been made in the analysis of laminated and anisotropic shells. Reviews of the many contributions on this subject are given in a number of survey papers (Ambartsumian, 1966, 1968; Bert and Egle, 1969; Bert and Francis, 1974; Bert, 1975; Grigolyuk and Kogan, 1972; Grigolyuk and Kulikov, 1988; Habip, 1965; Kapania, 1989; Pelekh, 1975; Teters, 1977) and monographs (see, for example, Alfutov *et al.*, 1984; Ambartsumian, 1974; Grigorenko and Vasilenko, 1981; Grigorenko *et al.*, 1987; Kovarik, 1985; Librescu, 1975; Pelekh and Lazko, 1982; Rasskazov *et al.*, 1986; Vanin and Semeniuk, 1987).

Most of the early publications on laminated cylindrical shells, and laminated shells in general, were limited to predicting the gross response characteristics (vibration frequencies, buckling loads, average through-the-thickness displacements and rotations) of thin shells. The classical laminated shell theory, incorporating the Kirchhoff–Love hypotheses of straight inextensional normals for the entire shell package (see Ambartsumian, 1966, 1974; Bert, 1975), is adequate for this purpose. The expanded use of fibrous composite materials in aircraft, automotive, shipbuilding and other industries has stimulated interest in the accurate prediction of the detailed response and failure characteristics of laminated anisotropic shells. Several modeling approaches have been proposed which take account of the relatively low ratio of the transverse shear modulus to the in-plane modulus in most of the advanced composites in use to date. Some of these modeling approaches are extensions of similar approaches used for isotropic shells, and include: (1) three-dimensional and quasi-three-dimensional elasticity models (Ahmed, 1966; Boresi, 1965; Chandrashekhara and Gopalakrishnan, 1982; Chou and Achenbach, 1972; Eason, 1963; Grigorenko and Vasilenko, 1981; Grigorenko *et al.*, 1984; Karlsson and Ball, 1966; Misovec and Kempner, 1970; Noor and Rarig, 1974; Noor and Peters, 1989a; Roy and Tsai, 1988; Srinivas, 1974); (2) first-order shear-deformation shell theories based on linear displacement and/or strain variation through the entire shell thickness (Bert and Kumar, 1982; Dong and Tso, 1972; Vasilenko and Golub, 1983); and (3) higher-order shear-deformation shell theories based on nonlinear (or piecewise linear) variation of displacements and/or stresses through the shell thickness (Barbero *et al.*, 1990; Bhimaraddi, 1985; Khdeir *et al.*, 1989; Librescu *et al.*, 1989; Narusberg and Pazhe, 1982; Reddy and Liu, 1985; Whitney and Sun, 1974).

In quasi-three-dimensional models simplifying assumptions are made regarding the stress (or strain) state in the shell (or in the individual layers), but no *a priori* assumptions are made about the distribution of the different response characteristics in the thickness direction. The use of both three-dimensional and quasi-three-dimensional models for predicting the response characteristics of multilayered cylinders, subjected to complicated loads, is computationally expensive, and therefore is not feasible for practical problems. Experience with two-dimensional shear-deformation theories has shown them to be inadequate for the accurate prediction of transverse stresses and deformations. This is particularly true when first-order theories [in which the transverse shear strains are assumed to be constant within each layer (Noor and Peters (1989b)] are used for analyzing medium-thick and thick cylinders.

A simple approach for the accurate evaluation of transverse stresses in composite cylinders is to use two-dimensional shear-deformation shell theory for calculating the in-plane stresses, and then apply the three-dimensional equilibrium equations to determine the transverse stresses. An improvement of this approach was proposed (Noor and Peters, 1989b) in which better estimates were predicted for the transverse shear stiffnesses and then used to correct the gross response characteristics.

Most of the literature on the accurate evaluation of detailed response characteristics of composite cylinders is limited to laminated cylinders with few layers which are rarely used in practice. The establishment of reliable and efficient modeling techniques for simulating the response of multilayered composite cylinders remains a challenging task and is the focus of the present study. Specifically, the objective of this paper is to assess the accuracy of a number of computational models, based on two-dimensional shear deformation theories, for multilayered composite cylinders. The composite cylinders considered herein consist of a number of perfectly bonded layers. The individual layers of the cylinder are assumed to be homogeneous and orthotropic. The sign convention for the different displacement and stress components are shown in Fig. 1.

The computational models considered in this study can be divided into three categories: global approximation models, discrete-layer models, and a predictor corrector approach. The three categories are described subsequently. Extensive numerical results are presented showing the effects of the different lamination and geometric parameters of the composite cylinder on the accuracy of the linear stress and free vibration responses obtained by the different models.

2. GLOBAL APPROXIMATION MODELS

Figure 1 shows the geometric characteristics of the multilayered cylinder as follows: L is length of the cylinder; r_0 is radius of the middle surface; and h is the total thickness of the cylinder. The dimensionless coordinates ξ, ζ are introduced, where:

$$\xi = \frac{x}{L}$$

$$\zeta = \frac{r-r_0}{h} = \frac{x_3}{h}. \tag{1}$$

2.1. *Kinematic assumptions*

The two-dimensional shear-deformation shell theories considered herein are based on the following displacement expansions in the thickness coordinate:

$$u = \sum_{i=0}^{J_u} u^{(i)}(x_3)^i$$

$$v = \sum_{i=0}^{J_v} v^{(i)}(x_3)^i$$

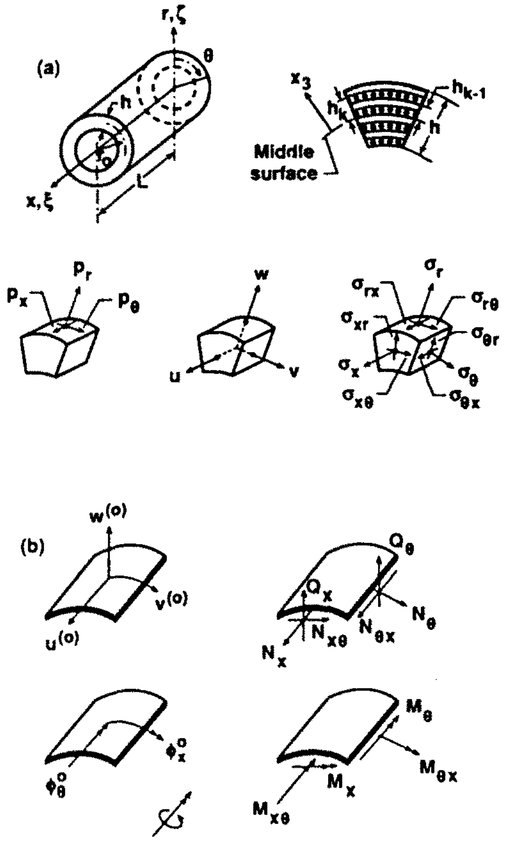

Fig. 1. Characteristics of laminated orthotropic cylinder and sign convention for stresses and displacements: (a) three-dimensional elasticity model; and (b) model 1 (based on first-order shear deformation theory).

$$w = \sum_{j=0}^{J_w} w^{(j)}(x_3)^j \tag{2}$$

where u, v, w are the displacement components in the x, θ, and r coordinate directions, respectively. The displacement parameters $u^{(j)}$, $v^{(j)}$ and $w^{(j)}$ are functions of x and θ only. In eqns (2), and henceforth, a superscript between parentheses does not refer to an exponent. The expressions of the strain components in terms of the displacement parameters are given in Appendix A.

2.2. *Displacement expansions*

For asymmetric response, each of the displacement parameters in eqns (2) is expanded in a double Fourier series in the ξ- and θ-directions such that the simply supported boundary conditions along the curved edges are satisfied. The following expansions are used:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^{(j)} = \sum_{m=1} \sum_{n=0} \begin{Bmatrix} u_{mn}^{(j)} \cos m\pi\xi \cos n\theta \\ v_{mn}^{(j)} \sin m\pi\xi \sin n\theta \\ w_{mn}^{(j)} \sin m\pi\xi \cos n\theta \end{Bmatrix}. \tag{3}$$

The external surface loads are also expanded in double Fourier series similar to the displacement components in their respective directions. For free vibration problems the right-hand sides of eqns (3) are multiplied by $e^{i\omega t}$, where ω is the frequency of vibration of the cylinder and t is time. Note that the displacement expansions, eqns (3), provide exact representations for the stress and free vibration responses of orthotropic cylinders.

2.3. *Governing equations*

The governing displacement equations of the cylinder are obtained by evaluating the potential and kinetic energies of the shell in terms of the displacement parameters [using eqns (A1)–(A5), Appendix A] and applying Hamilton's principle (or, for static loading, the principle of minimum potential energy). Exact integration is performed in the thickness direction. For simply supported orthotropic cylinders the governing equations uncouple in harmonics. For each pair of harmonics, m and n, the governing displacement equations can be written in the following compact form:

$$[K]_{mn}\{X\}_{mn} = \{P\}_{mn} + \omega_{mn}^2[M]\{X\}_{mn} \tag{4}$$

where $\{X\}_{mn}$ is the vector of unknown displacement parameters, $u_{mn}^{(i)}$, $v_{mn}^{(i)}$ and $w_{mn}^{(i)}$; $[K]_{mn}$ and $[M]$ are the stiffness and mass matrices of the cylinder; $\{P\}_{mn}$ is the vector of external loading. For static loading problems $\omega_{mn} = 0$, and for free vibration problems $\{P\}_{mn} = 0$.

3. DISCRETE-LAYER MODELS

The discrete-layer theories considered herein are based on the following piecewise linear displacement approximations in the thickness coordinate (see, for example, Barbero *et al.*, 1990):

$$u = u^{(0)} + \sum_{i=1}^{k} u^{(i)}\psi(k,i)$$

$$v = v^{(0)} + \sum_{i=1}^{k} v^{(i)}\psi(k,i)$$

$$w = w^{(0)} \tag{5}$$

where $\psi(k,i)$ are piecewise linear functions in x_3, given by:

$$\psi(k,i) = \frac{x_3 - h_{k-1}}{h_k - h_{k-1}}, \quad i = k$$

$$= 1, \qquad i \neq k.$$

For free vibration problems the right-hand sides of eqns (5) are multiplied by $e^{i\omega t}$. The total number of displacement parameters equals $2NL+3$, where NL is the total number of layers in the cylinder. The strain displacement relations *in each layer* are taken to be the same as those of the first-order shear deformation theory (see Dong and Tso, 1972; Noor and Peters, 1989b). A simplified discrete layer model is obtained by imposing the continuity of the transverse shear stresses σ_{rx} and $\sigma_{r\theta}$, at the interfaces between layers. The number of displacement parameters is then reduced to five (as in the first-order shear deformation shell theory—see, for example, DiScuiva, 1987).

4. PREDICTOR CORRECTOR APPROACH

It has long been recognized that the range of validity of the first-order shear-deformation shell theory is strongly dependent on the factors used in adjusting the transverse shear stiffnesses of the cylinder. Several approaches have been proposed for calculating the composite shear correction factors for different laminates. Most of these approaches are based on matching certain gross response characteristics, as predicted by the first-order theory, with the corresponding characteristics of the three-dimensional elasticity theory. Among the gross response characteristics are transverse shear strain energy, natural frequency associated with the thickness shear vibration mode, and velocity of propagation of a flexural wave (see, for example, Chow, 1971; Whitney, 1973). However, all the shear correction factors proposed in the cited references are calculated *a priori* and are therefore functions of the lamination parameters only. They do not account for the differences in the distribution of the transverse shear strains in the thickness direction resulting from different loading conditions. As an attempt to incorporate the actual distribution of the transverse shear strains in the thickness direction of the cylinder, in calculating the transverse shear stiffnesses, a predictor–corrector approach was proposed in Noor and Peters (1989b) for the *a posteriori* determination of accurate shear correction factors and for adjusting the transverse shear stiffnesses of the multilayered cylinder. The approach is highlighted herein.

The predictor phase consists of using a first-order shear-deformation theory (see Appendix A) to calculate the initial estimates for the gross response characteristics of the cylinder (vibration frequencies, average through-the-thickness displacements and rotations), as well as the in-plane stresses. Then, three-dimensional equilibrium equations and constitutive relations are used to calculate: (a) transverse shear and transverse normal stresses and strains; (b) through-the-thickness strain energy density distributions; and (c) accurate *a posteriori* estimates for the composite shear correction factors. The estimates of the composite shear correction factors are obtained by matching the integral of the transverse shear strain energy in the thickness direction with that obtained from the first-order theory (see Noor and Peters, 1989b). These composite correction factors are used to adjust the transverse shear stiffnesses of the cylinder. The corrector phase consists of using the adjusted transverse shear stiffnesses, in conjunction with a reanalysis procedure, to obtain better estimates for the gross response characteristics, as well as for the distributions of displacements and in-plane stresses in the thickness direction. The effectiveness of this two-phase procedure is demonstrated in the section on numerical studies.

5. THREE-DIMENSIONAL MODELS

In order to assess the accuracy of the predictions of the different two-dimensional models, exact three-dimensional elasticity solutions are obtained for multilayered composite cylinders. The cylinders are assumed to be orthotropic and are simply supported along the curved edges.

Each of the displacement and stress components is expanded in a double-Fourier series in the ξ- and θ-directions such that the boundary conditions along the curved edges are satisfied. The governing equations of the cylinder are thereby reduced to simultaneous ordinary differential equations which uncouple in harmonics. For each pair of harmonics, a Frobenius-type method is applied for the solution of the ordinary differential equations. The method is described in detail in Srinivas (1974).

6. NUMERICAL STUDIES

The accuracy of the stress and vibrational responses of multilayered cylinders predicted by different two-dimensional models is strongly dependent on the significance of the transverse shear deformations which, in turn, depends on a number of parameters including:

(a) lamination parameters (namely, number of layers, stacking sequence, degree of orthotropy, and fiber orientation of the different layers);

(b) geometric parameters (e.g., thickness-to-radius and length-to-radius ratios);

(c) type and rate of variation of external loading (e.g., longitudinal and circumferential wave numbers); and

(d) boundary (or support) conditions.

Due to the large number of these parameters and the fact that closed form (or analytic) solutions are only obtainable for cylinders with simple geometries (e.g., circular profile and constant stiffness), loading and boundary conditions, it is impractical to present quantitative results of a general nature. Several numerical studies have been made of the accuracy of the static and free vibrational responses predicted by different two-dimensional models (see, for example, Grigorenko and Vasilenko, 1981; Khdeir *et al.*, 1989; Librescu *et al.*, 1989). However, most of these studies were for laminated cylinders with a small number of layers.

Herein, the results of parametric studies for multilayered composite cylinders are presented. These studies were conducted to provide some insight into the effects of variation in the lamination and geometric parameters of multilayered composite cylinders on the accuracy of the response characteristics predicted by eight different modeling approaches based on two-dimensional shear-deformation laminated shell theories. The modeling approaches considered are listed in Table 1, and will henceforth be referred to as models 1–8.

The composite cylinders considered in the present study are simply supported laminated circular cylinders. The fibers of the different layers alternate between the circumferential and longitudinal directions, with the fibers of the top layer running in the circumferential direction. The total thickness of the circumferential and longitudinal layers in each shell was the same. The material characteristics of the individual layers were taken to be those typical of high-modulus fibrous composites, namely:

$$G_{LT}/E_T = 0.5, \quad G_{TT}/E_T = 0.3356, \quad \nu_{LT} = 0.3, \quad \nu_{TT} = 0.49$$

where subscript L refers to direction of fibers and subscript T refers to the transverse direction; ν_{LT} is the major Poisson's ratio. For static stress analysis problems, the cylinders were subjected to internal normal loading of the form: $p_r = p_0 \sin \pi\xi \cos n\theta$; for free vibration problems, only the lowest frequencies for each pair of m, n and the associated mode shapes and modal stresses were considered. For each problem, the solutions obtained by the aforementioned modeling approaches were compared with exact three-dimensional elasticity solutions.

Six parameters were varied, namely, the number of layers, NL; the degree of orthotropy of the individual layers, E_L/E_T; the thickness-to-radius ratio, h/r_0; the length-to-radius ratio of the cylinder, L/r_0; the longitudinal and circumferential wave numbers, m and n. The

Table 1. Modeling approaches used in the numerical studies

Model no.	Description	Through-the-thickness displacement assumptions	Constraint conditions on stresses	Total number of displacement parameters
1, 1A	First-order shear-deformation theory	linear u, v constant w	$\sigma_z = 0$	5
2	First-order theory with transverse normal stresses and strains included	linear u, v and w	none	6
3	Lo-Christensen Wu type theory	cubic u, v quadratic w	none	11
4	Higher-order shear deformation theory	quintic u, v and w	none	18
5	Simplified higher-order theory	cubic u, v constant w	$\sigma_z = 0$ throughout and σ_{xz} and $\sigma_{\theta z} = 0$ at top and bottom surfaces	5
6	Discrete layer theory	piecewise linear u, v constant w (through-the thickness)	$\sigma_z = 0$ throughout	$2NL+3$
7	Simplified discrete layer theory	piecewise linear u, v constant w (through-the-thickness)	$\sigma_z = 0$ continuity of σ_{xz} and $\sigma_{\theta z}$ at layer interfaces	5
8	Predictor–corrector approach (see Noor and Peters, 1989b)	Predictor phase linear u, v constant w Corrector phase matching displacements	Predictor phase $\sigma_z = 0$ Corrector phase none	5

In model 1, $k_x = k_\theta = 1$, and in model 1A, they are computed from the cylindrical bending condition of Chow (1971) and Whitney (1973).

number of layers was varied between 2 and 20, E_L/E_T between 3 and 30, h/r_0 between 0.01 and 0.3, L/r_0 between 0.5 and 5.0, m between 1 and 3, and n betweeen 0 and 8. The assessment of the accuracy of the eight modeling approaches listed in Table 1 included both global response characteristics (vibration frequencies and strain energy components), as well as detailed stress and displacement distributions in the thickness direction.

The effects of variation in the six parameters NL, E_L/E_T, h/r_0, L/r_0, m and n on the minimum vibration frequencies, and the energy components U_{be}, U_{sh} and U_{tn} (see Appendix A), obtained by the three-dimensional model, are depicted in Figs 2, 3 and 4. As can be seen from Figs 3 and 4, the transverse shear strain energy ratio, U_{sh}/U, increases with the increase in NL, E_L/E_T, h/r_0 and n. Even for cylinders with $h/r_0 = 0.1$, U_{sh}/U can exceed 0.2 (for $NL = 10$, $E_L/E_T = 15$, $L/r_0 = 1$, and $n > 4$). The increase in U_{sh}/U is associated with a decrease in the ratio U_{be}/U. On the other hand, for all the vibration problems considered, U_{tn}/U was found to be very small (less than 1%). For statically loaded cylinders U_{tn}/U approaches 37% for thick multilayered cylinders with $h/r_0 = 0.3$, $L/r_0 = 1$, $E_L/E_T = 15$, $n \geq 8$, and $NL \geq 10$ (Fig. 4).

An indication of the accuracy of the minimum vibration frequencies and the total strain energy predicted by the models listed in Table 1 is given in Figs 5, 6 and 7, and Table 2. Figure 8 shows the effect of h/r_0 on the distribution of displacements, stresses, and transverse shear strain energy density in the thickness direction, obtained by the three-dimensional elasticity model. As can be seen from Fig. 8, the distribution of σ_{rx} and U_{rx} is fairly insensitive to variations in h/r_0. An indication of the accuracy of displacement, stress,

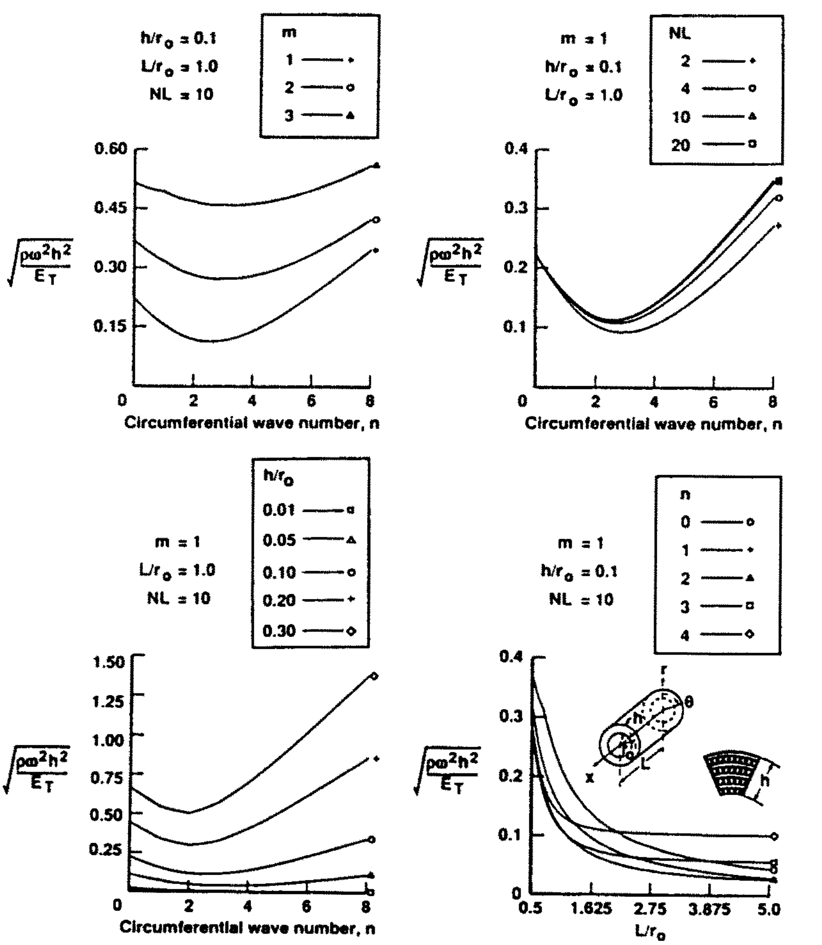

Fig. 2. Effect of lamination and geometric parameters on minimum frequencies of vibration predicted by the three-dimensional elasticity model. Simply supported composite cylinders with $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$ and $\nu_{TT} = 0.49$.

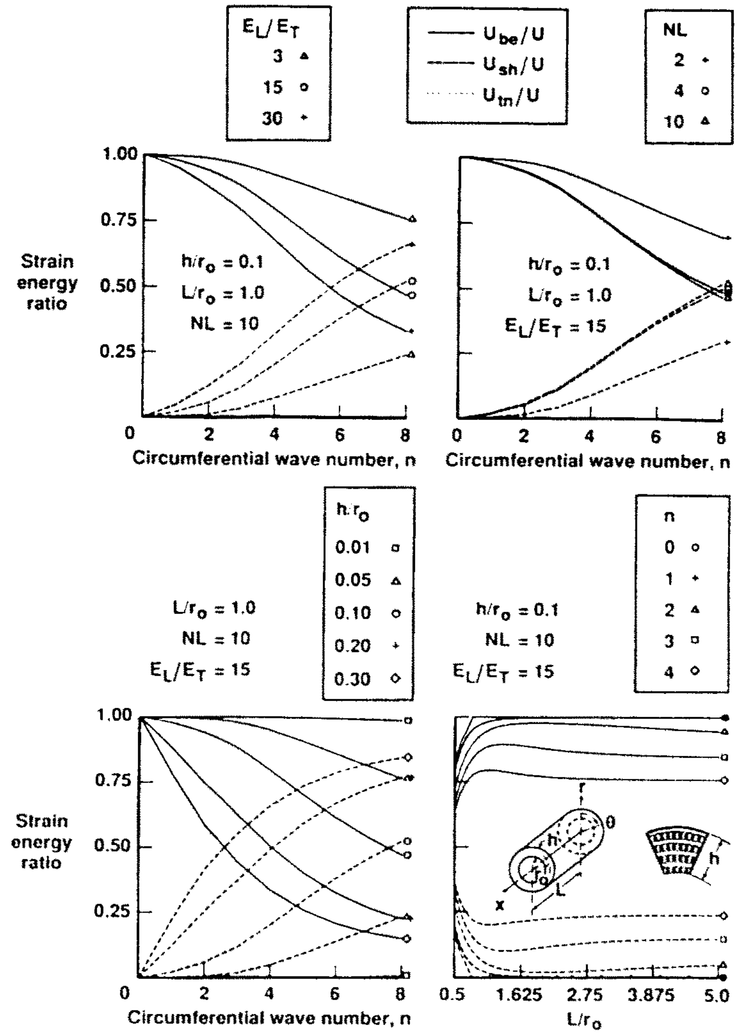

Fig. 3. Effect of lamination and geometric parameters on strain energy components, associated with minimum vibration frequencies obtained by three-dimensional elasticity model. Simply supported composite cylinders with $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$ and $m = 1$.

and transverse shear strain energy distributions predicted by models 3–8 for multilayered cylinders is given in Figs 9 and 10 for the free vibration case, and in Figs 11 and 12 for the static loading case. Each of the response quantities in Figs 8–12 is normalized by its maximum absolute value, obtained by the three-dimensional elasticity model. An indication of the relative magnitudes of the different displacement and stress components shown in Figs 8–12 is provided by the ratios of their maximum values given in Table 3. An examination of the numerical results reveals the following.

(1) As expected, the accuracy of the first-order shear-deformation theory (model 1) decreases as h/r_0 and n increase (see Table 2 and Figs 5–7). The range of validity of the first-order theory is strongly dependent on the values of the composite shear correction factors used, k_x and k_θ. For free vibration problems when k_x and k_θ were selected to be 1, the error in the minimum frequency for cylinders with $h/r_0 = 0.05$, $L/r_0 = 1$, and $n = 8$ is 2.8%. As h/r_0 increases to 0.2, the error increases to 9.2%. When k_x and k_θ were computed from the cylindrical bending condition of Chow (1971) and Whitney (1973), the corresponding errors were less than 0.01% and 1.2% (see Table 2).

(2) Despite the larger number of displacements of model 2, its predictions are generally less accurate than those of model 1 (see Figs 5 and 6). This is attributed to the assumption of constant transverse normal strain, and piecewise constant transverse normal stresses,

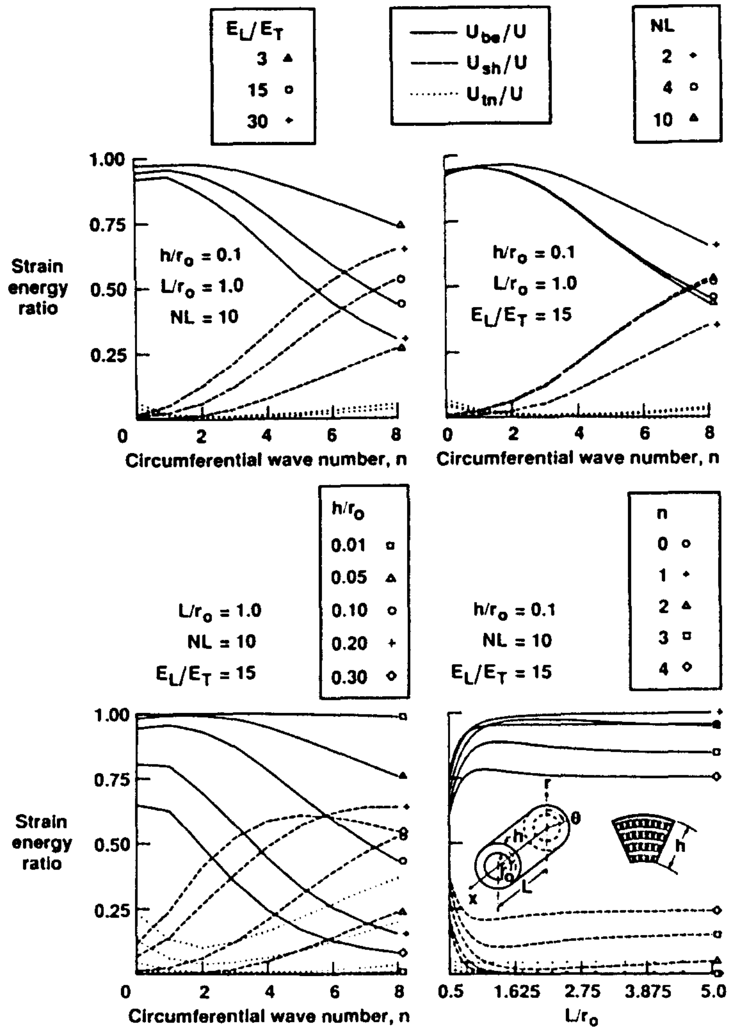

Fig. 4. Effect of lamination and geometric parameters on strain energy components. Simply supported composite cylinders subjected to internal pressure $p_r = p_0 \sin \pi\xi \cos n\theta$. $G_{LT}/E_T = 0.5$. $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$ and $m = 1$.

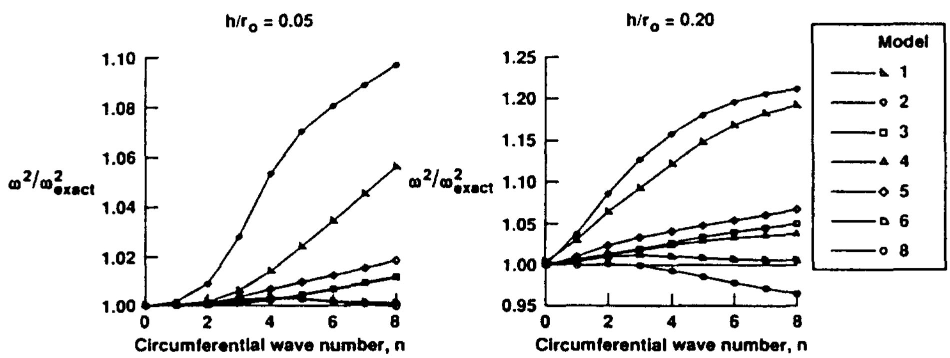

Fig. 5. Effect of thickness ratio, h/r_0, and circumferential wave number, n, on the accuracy of the minimum vibration frequencies obtained by different models (see Table 1). Simply supported composite cylinders with $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$, $L/r_0 = 1.0$ and $m = 1$.

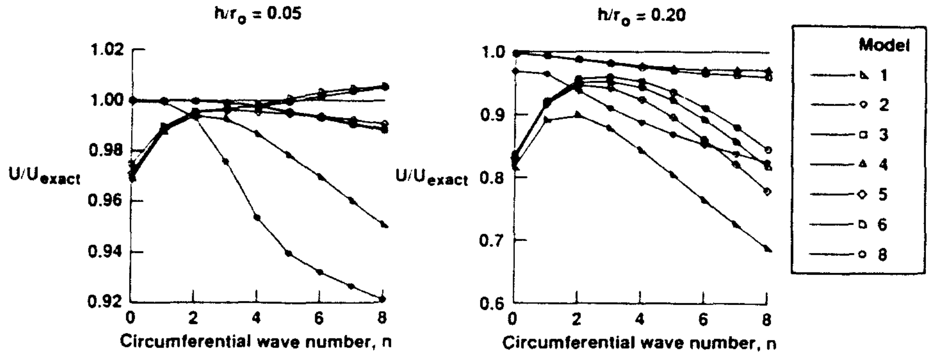

Fig. 6. Effect of thickness ratio, h/r_0, and circumferential wave number, n, on the accuracy of the total strain energy obtained by different models (see Table 1). Simply supported composite cylinders subjected to internal pressure $p_r = p_0 \sin \pi\xi \cos n\theta$, $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$ and $L/r_0 = 1.0$.

through the thickness in model 2, which results in considerably overestimating the in-plane stresses σ_x and σ_θ. An exception to that is the case of statically loaded thick cylinders (with $h/r_0 \geq 0.2$, see Fig. 6).

(3) The global response characteristics predicted by the higher-order shear-deformation theories (models 3 and 4) are fairly accurate. For multilayered cylinders with $h/r_0 \leq 0.2$ and $n \leq 8$, the maximum errors in the minimum frequency of vibration were less than 2.5%. Both models 3 and 4 slightly overestimate the vibration frequencies (see Fig. 5). For the static loading case, they underestimate the total strain energy U (see Fig. 6). The small differences between the predictions of models 3 and 4 point to the slow convergence of the displacement expansions used, eqns (2). The distribution of the transverse stresses through the thickness, obtained by models 3 and 4, is not as accurate as the gross response characteristics (see Figs 9 and 11). This is particularly true for the transverse shear stresses, σ_{rx}, and the transverse shear strain energy density, U_{rx}, obtained by models 3 and 4; as well as the modal transverse normal stresses, σ_r, predicted by model 3 (see Fig. 9).

(4) The predictions of the simplified higher-order theory (model 5) are fairly accurate. For multilayered cylinders with $h/r_0 \leq 0.2$ and $n \leq 8$, the error in the minimum frequency is less than 3.4%. A rapid degradation in accuracy occurs in cylinders with $h/r_0 \geq 0.2$, as the circumferential wave number increases beyond 4. Model 5 overestimates the vibration frequencies and underestimates the total strain energy (see Figs 5 and 6). The in-plane displacements, u and v, predicted by this model are fairly accurate. However, the transverse shear stresses, σ_{rx}, predicted by this model are grossly in error. Moreover, the model does not predict the transverse normal stresses, σ_r (see Figs 10 and 12).

(5) The global response characteristics predicted by the discrete-layer theory (model 6) are very accurate. The maximum errors in the minimum frequency of vibration were less

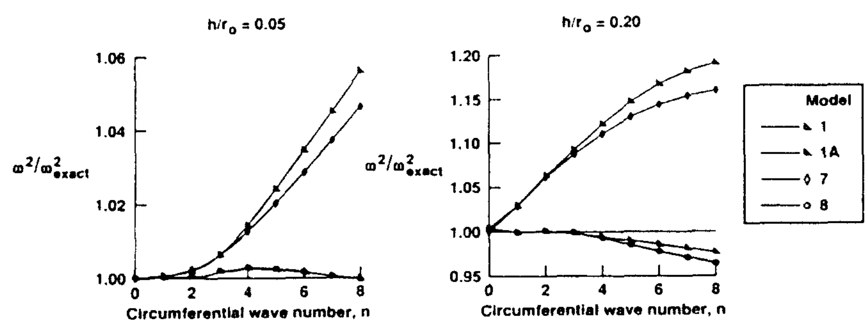

Fig. 7. Effect of thickness ratio, h/r_0, and circumferential wave number, n, on the accuracy of the minimum vibration frequencies obtained by models 1, 1A, 7 and 8 (see Table 1). Simply supported composite cylinders with $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$, $L/r_0 = 1.0$ and $m = 1$

Table 2. Effect of composite correction factors and reanalysis procedure on the accuracy of the lowest vibration frequencies obtained by models 1, 1A and 8 (simply supported composite cylinders with $NL = 10$, $E_L/E_T = 15$, $L/r_0 = 1.0$ and $m = 1$)

			Values of $\omega^2/\omega^2_{exact}$					
			Model		Model 8			
n	Ω_{exact}	$U_{sh}/U \times 10^2$	1	1A	Taylor series	Full reanalysis	k_x	k_θ
(a) $h/r_0 = 0.05$, $n = 3$								
0	1.234	0	1.000	1.000	1.000	1.000	1.0	0.8015
1	0.5447	0.1950	1.001	1.000	1.000	1.000	0.7545	0.9154
2	0.2557	0.6523	1.002	1.001	1.001	1.001	0.7521	0.8649
3	0.1753	1.898	1.006	1.002	1.003	1.002	0.7556	0.8025
4	0.1938	4.910	1.014	1.003	1.005	1.003	0.7610	0.7834
5	0.2987	9.267	1.024	1.002	1.007	1.003	0.7667	0.7768
6	0.5036	13.95	1.035	1.002	1.008	1.002	0.7718	0.7740
7	0.8288	18.65	1.045	1.001	1.009	1.001	0.7762	0.7725
8	1.295	23.30	1.056	1.000	1.010	1.000	0.7799	0.7716
(b) $h/r_0 = 0.20$, $n = 2$								
0	19.74	0	1.004	1.004	1.004	1.004	1.0	0.8146
1	11.77	12.17	1.030	0.999	1.004	0.999	0.7705	1.274
2	8.933	25.48	1.064	1.000	1.010	1.001	0.7692	0.8325
3	10.84	37.34	1.092	0.998	1.012	0.999	0.7703	0.7828
4	16.65	49.25	1.122	0.994	1.009	0.993	0.7720	0.7701
5	26.06	59.45	1.148	0.990	1.002	0.985	0.7739	0.7650
6	38.88	67.23	1.168	0.986	0.994	0.978	0.7760	0.7624
7	54.95	72.95	1.182	0.981	0.986	0.971	0.7784	0.7609
8	74.18	77.14	1.191	0.977	0.978	0.965	0.7809	0.7599

$\Omega_{exact} = 10^2 \times \rho h^2 \omega^2_{exact}/E_T$. In model 1, $k_x = k_\theta = 1$ and in model 1A, $k_x = k_\theta = 0.7731$, as computed from the cylindrical bending condition of Chow (1971) and Whitney (1973).

than 0.6%. For cylinders with $h/r_0 \leq 0.1$, the total strain energies predicted by this model are similar to those predicted by model 4 (see Fig. 6). As h/r_0 increases the accuracy of the total strain energy predicted by this model decreases. The distributions of in-plane stresses, transverse shear stresses, and displacements through the thickness obtained by model 6 are fairly accurate (see Figs 9 and 11). However, the model does not predict transverse normal stresses, σ_r. Note that for $NL \geq 8$ the number of displacement parameters used in this model exceeds those used in all other models.

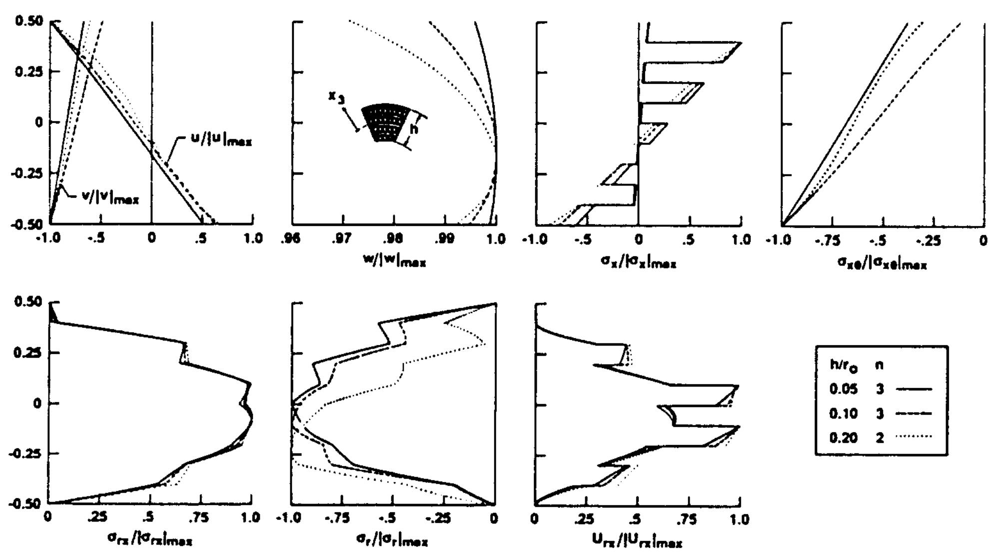

Fig. 8. Effect of the thickness ratio, h/r_0, on the distribution in the thickness direction, of displacements, stresses and transverse shear strain energy density, associated with minimum vibration frequencies. Simply supported composite cylinders with $L/r_0 = 1.0$, $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$ and $m = 1$.

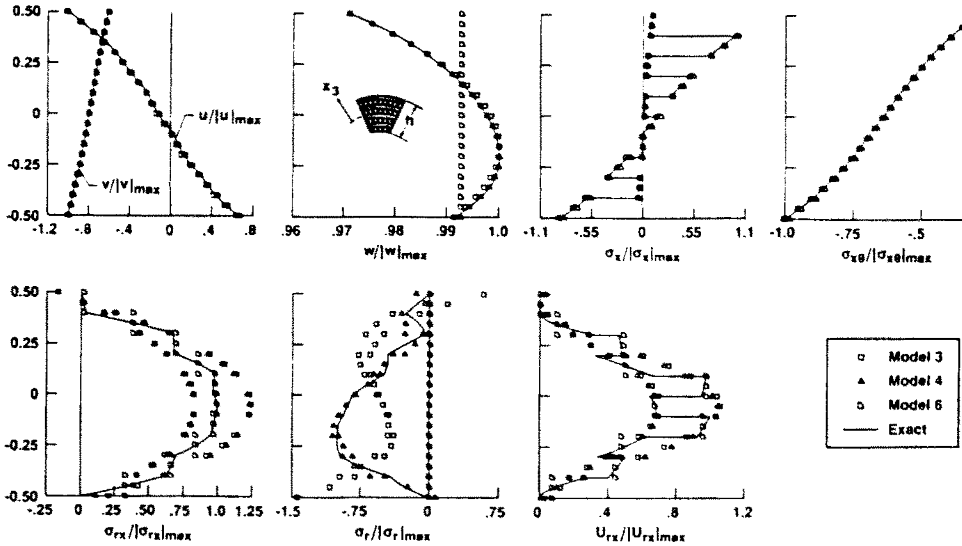

Fig. 9. Accuracy of displacements, stresses, and transverse shear strain energy density, associated with minimum frequency obtained by models 3, 4 and 6 (see Table 1). Simply supported composite cylinders with $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$, $h/r_0 = 0.2$, $L/r_0 = 1.0$, $m = 1$ and $n = 2$.

(6) The accuracy of the predictions of the simplified discrete-layer theory, model 7, is comparable to that of the first-order shear deformation theory with the same number of displacement parameters, model 1. This is true for both the global as well as detailed response characteristics.

(7) The predictor corrector approach (model 8) appears to be a very effective procedure for the accurate determination of the global, as well as the detailed response characteristics of cylinders. Specifically, the following four observations can be noted:

(a) The numerical values of the corrected composite shear correction factors, k_x and k_θ, are fairly insensitive to their initial values, k_x^0 and k_θ^0, used in the first-order shear-deformation theory. They depend on the distributions of the transverse shear strains in the

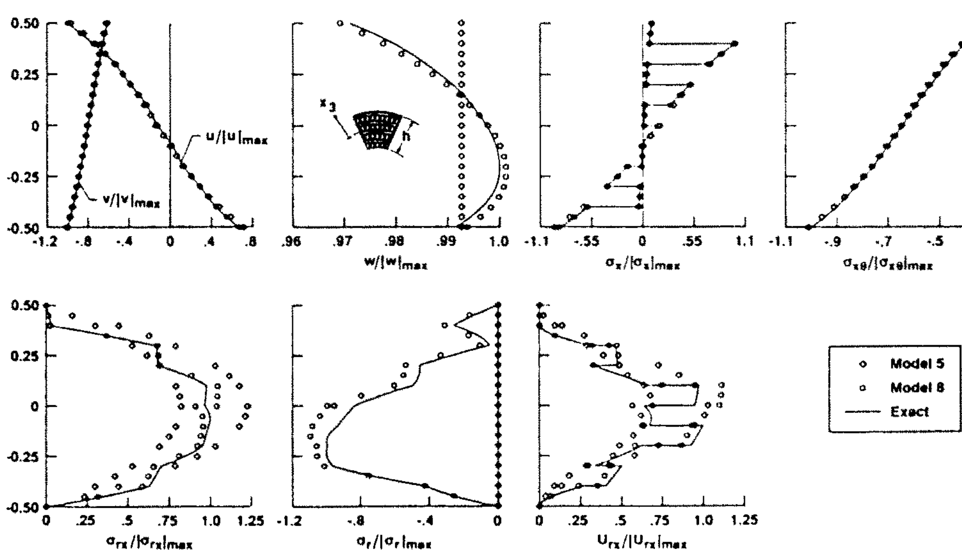

Fig. 10. Accuracy of displacements, stresses, and transverse shear strain energy density, associated with minimum frequency obtained by models 5 and 8 (see Table 1). Simply supported composite cylinders with $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$, $h/r_0 = 0.2$, $L/r_0 = 1.0$, $m = 1$ and $n = 2$.

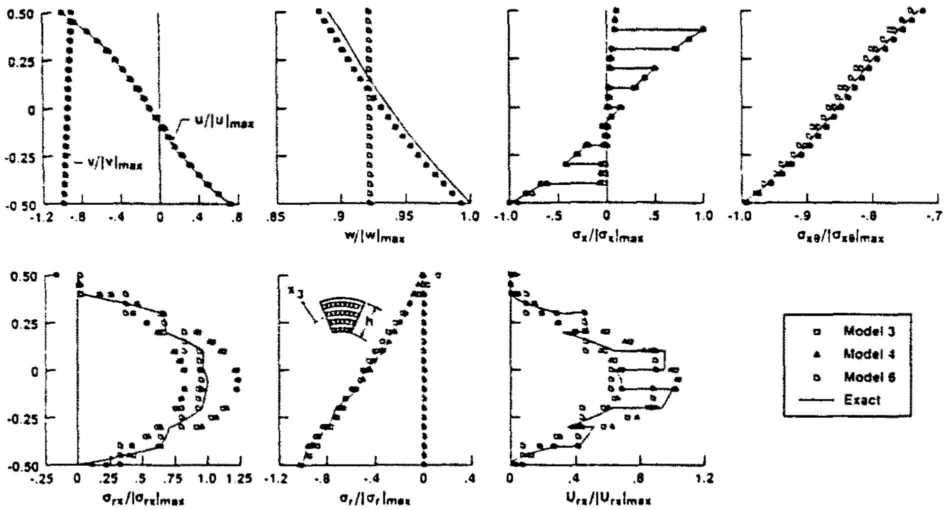

Fig. 11. Accuracy of displacements, stresses, and transverse shear strain energy density obtained by models 3, 4 and 6 (see Table 1). Simply supported composite cylinder subjected to internal pressure $p_r = p_0 \sin \pi\xi \cos \theta$, $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$, $h/r_0 = 0.2$ and $L/r_0 = 1.0$.

thickness direction which, in turn, are functions of both the lamination and geometric parameters of the cylinder.

(b) If k_x^0 and k_θ^0 are both selected to be 1, the error in the global response quantities obtained in the first (predictor) phase, for shells with $NL = 10$, $h/r_0 \geq 0.2$, and $n \geq 4$, may be unacceptable; however, the corrector phase improves these predictions substantially, and results in highly accurate distributions of displacements and stresses through the thickness (see Figs 7, 10 and 12).

(c) The accuracy of the response quantities obtained using the predictor–corrector approach is insensitive to the initial shear correction factors selected. It is also insensitive to the selection of the reanalysis procedure in the correction phase. For example, when the

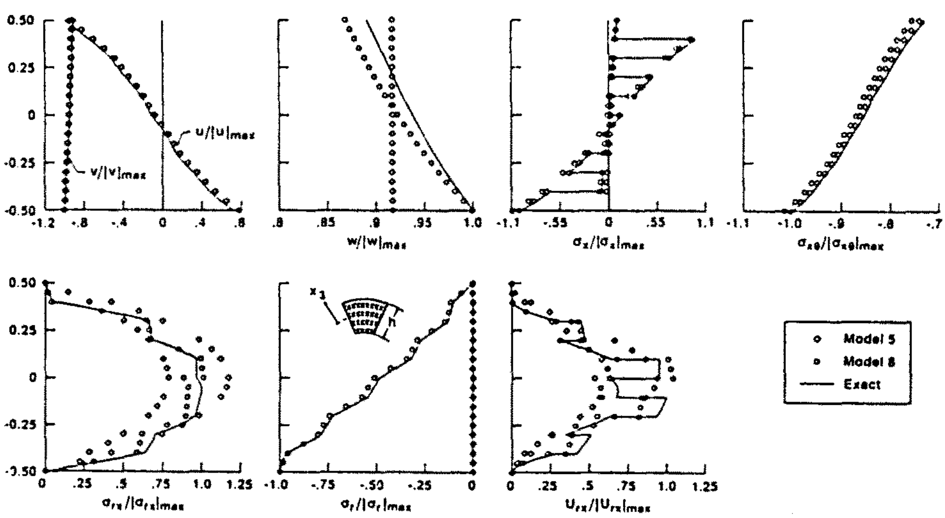

Fig. 12. Accuracy of displacements, stresses, and transverse shear strain energy density obtained by models 5 and 8 (see Table 1). Simply supported composite cylinder subjected to internal pressure $p_r = p_0 \sin \pi\xi \cos \theta$, $NL = 10$, $E_L/E_T = 15$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.3356$, $\nu_{LT} = 0.3$, $\nu_{TT} = 0.49$, $h/r_0 = 0.2$ and $L/r_0 = 1.0$.

Table 3. Relative magnitudes of the maximum displacements and stresses obtained by the three-dimensional elasticity model (simply supported composite cylinders with $NL = 10$, $E_l/E_T = 15$, $L/r_0 = 1.0$ and $m = 1$)

	Free vibrational response			Static response
	$h/r_0 = 0.05$ $n = 3$ (Fig. 8)	$h/r_0 = 0.10$ $n = 3$ (Fig. 8)	$h/r_0 = 0.2$ $n = 2$ (Figs 8–10)	$h/r_0 = 0.2$ $n = 1$ (Figs 11 and 12)
$\lvert u\rvert_{max}/\lvert w\rvert_{max}$	0.099	0.165	0.240	0.218
$\lvert v\rvert_{max}/\lvert w\rvert_{max}$	0.386	0.444	0.589	0.604
$\lvert\sigma_{x\theta}\rvert_{max}/\lvert\sigma_x\rvert_{max}$	0.171	0.135	0.129	0.134
$\lvert\sigma_{rx}\rvert_{max}/\lvert\sigma_x\rvert_{max}$	0.020	0.043	0.080	0.084
$\lvert\sigma_r\rvert_{max}/\lvert\sigma_x\rvert_{max}$	0.005	0.011	0.010	0.119

calculated composite correction factors are much different from their initial values, the first-order Taylor series approximation (with respect to the composite correction factors) provides sufficiently accurate estimates for the response quantities (see Table 2).

(d) Because of the assumed through-the-thickness linear distribution of strains in the predictor phase, and the associated piecewise linear distribution of stresses, the stress conditions at the top and bottom surfaces, and at layer interfaces, *cannot all be satisfied simultaneously*. The accuracy of the transverse stresses obtained by the predictor corrector approach was found to be somewhat sensitive to which conditions are satisfied. Numerical experiments have shown that good accuracy is obtained when the stress conditions at both the top and bottom surfaces are satisfied, and the discontinuities in the transverse stresses occur at or near the middle surface. The stress discontinuities can be reduced by using an error distribution procedure. Such a procedure was not used in the present study.

The aforementioned observations point to the fact that accurate prediction of the distribution of stresses and displacements through-the-thickness of multilayered cylinders requires the use of three-dimensional equilibrium and constitutive relations. These equations can be used in an inexpensive, postprocessing mode with any of the modeling approaches based on two-dimensional theories. The predictor corrector approach has the advantage of starting with a simple first-order theory in the first phase to obtain estimates for the global response characteristics, and then correcting these estimates before calculating the displacement distribution in the thickness direction.

7. POTENTIAL OF THE PREDICTOR CORRECTOR MODELING APPROACH

The predictor corrector approach appears to have high potential for the accurate prediction of vibration frequencies, stresses and deformations in multilayered composite cylinders. The numerical studies conducted for simply supported laminated orthotropic cylinders demonstrated the accuracy and effectiveness of this modeling approach. In particular, the following two points are worth mentioning:

(1) The predictor corrector approach can be applied, in conjunction with finite element models, to the analysis of anisotropic shells with arbitrary geometry. The calculation of the transverse stresses, composite shear correction factors, and the correction phase can be performed on the element level for selected elements (in the critical regions of the shell model).

(2) Although any of the two-dimensional shear-deformation shell theories can be used in the first (predictor) phase of the predictor-corrector approach, the first-order shear-deformation shell theory has two major advantages over other theories of: (a) only five displacement parameters are used to describe the deformation; and (b) in the finite element implementation only C^0 continuity is required. The simplified higher-order shell theory and the simplified discrete-layer theory, models 5 and 7, share the first advantage, but require C^1 continuity in their finite element implementation.

8. CONCLUDING REMARKS

A study is made of the effects of variation in the lamination and geometric parameters of multilayered composite cylinders on the accuracy of the static and vibrational responses predicted by eight modeling approaches, based on two-dimensional shear-deformation theories. The first seven modeling approaches considered are: first-order shear-deformation theory (based on linear variation of u, v and constant w, through-the-thickness); first-order theory with linear variation of u, v and w through-the-thickness; two higher-order theories (based on cubic and quintic variations for u, v and w through-the-thickness); a simplified higher-order theory (based on cubic variations of u, v through-the-thickness, but imposing the transverse shear stress conditions at the top and bottom surfaces of the cylinder); discrete-layer theory (with piecewise linear variation of the in-plane displacements in the thickness direction); and a simplified discrete-layer theory with the continuity of transverse stresses imposed at layer interfaces to reduce the number of generalized displacement parameters to five. The eighth model is a predictor–corrector approach based on using a first-order shear deformation theory to predict the generalized displacements, in-plane strains and stresses in the plate; and using the equilibrium equations and constitutive relations of the three-dimensional theory of elasticity to: (a) calculate the transverse stresses, strains and strain energy distribution in the different layers; and (b) provide accurate estimates for the composite shear correction factors and adjust the transverse shear stiffnesses. The adjusted stiffnesses are used, in conjunction with a reanalysis technique, to obtain corrected estimates for the different response quantities. The potential of the predictor–corrector approach for the accurate determination of the response characteristics of multilayered shells with complicated geometry is also discussed.

Extensive numerical results are presented for simply supported laminated orthotropic circular cylinders. Two key elements distinguish the present study from previous studies reported in the literature: (a) the standard of comparison is taken to be the exact three-dimensional elasticity solutions; and (b) quantities compared are not limited to gross response characteristics (e.g., vibration frequencies, strain energy components, average through-the-thickness displacements and rotations), but include detailed, through-the-thickness distributions of displacements, stresses and strain energy densities.

Based on the numerical studies conducted, the following conclusions seem to be justified.

(1) For most practical problems, the transverse shear deformation has a much more pronounced effect on the response of multilayered composite cylinders than that of transverse normal strain and stress. The latter can only become noticeable (of the order of 20% or more) for statically loaded thick cylinders and deformations with very short wavelength ($h/r_0 \geqslant 0.2$ and $n \geqslant 8$), and in the regions of highly localized loadings (or loadings with sharp variations).

(2) The accuracy of the predictions of first-order shear-deformation theory is strongly dependent on the values of the composite shear correction factors used. The use of the composite shear correction factors proposed in Chow (1971) and Whitney (1973) results in fairly accurate gross response characteristics for a wide range of lamination and geometric parameters.

(3) The accurate prediction of the stress and displacement distribution through-the-thickness of multilayered cylinders requires the use of three-dimensional equilibrium and constitutive relations. These equations can be used in an inexpensive, postprocessing mode with any of the modeling approaches based on two-dimensional theories.

(4) The predictor–corrector approach appears to be a very effective procedure for the accurate determination of the global as well as the detailed response characteristics of multilayered cylinders. The accuracy of the response quantities obtained in the first (predictor) phase for cylinders with a thickness-to-radius ratio of the order of 0.2 may be unacceptable. However, the corrector phase improves the predictions substantially and results in highly accurate distributions of displacements and stresses through the thickness.

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APPENDIX A: FUNDAMENTAL EQUATIONS OF THE TWO-DIMENSIONAL SHEAR-DEFORMATION THEORIES USED IN THE PRESENT STUDY

The fundamental equations of the higher-order shear deformation theories used in the present study as given in this Appendix.

Strain displacement relations

The strain components can be expressed in terms of the displacement parameters of eqns (2) as follows:

$$\varepsilon_x = \sum_{j=0}^{J_u} \partial_x u^{(j)}(x_3)^j$$

$$\varepsilon_\theta = \frac{1}{r}\left[\sum_{j=0}^{J_v} \partial_\theta v^{(j)}(x_3)^j + \sum_{j=0}^{J_w} w^{(j)}(x_3)^j\right]$$

$$\varepsilon_r = \sum_{j=0}^{J_w} j w^{(j)}(x_3)^{j-1}$$

$$\gamma_{r\theta} = \sum_{j=0}^{J_v} v^{(j)}\left(j - \frac{x_3}{r}\right)(x_3)^{j-1} + \frac{1}{r}\sum_{j=0}^{J_w} \partial_\theta w^{(j)}(x_3)^j$$

$$\gamma_{rx} = \sum_{j=0}^{J_u} j u^{(j)}(x_3)^{j-1} + \sum_{j=0}^{J_w} \partial_x w^{(j)}(x_3)^j$$

$$\gamma_{x\theta} = \frac{1}{r}\sum_{j=0}^{J_u} \partial_\theta u^{(j)}(x_3)^j + \sum_{j=0}^{J_v} \partial_x v^{(j)}(x_3)^j$$

where

$$\partial_x \equiv \frac{\partial}{\partial x}, \quad \partial_\theta \equiv \frac{\partial}{\partial \theta} \quad \text{and} \quad r = r_0\left(1 + \frac{x_3}{r_0}\right). \tag{A1}$$

Constitutive relations

The stress strain relations for a typical orthotropic layer, k, of the cylinder are given by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \sigma_{r\theta} \\ \sigma_{rx} \\ \sigma_{x\theta} \end{Bmatrix}^{(k)} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot \\ & c_{22} & c_{23} & \cdot & \cdot & \cdot \\ & & c_{33} & \cdot & \cdot & \cdot \\ & & & c_{44} & \cdot & \cdot \\ & & & & c_{55} & \cdot \\ & & & & & c_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{r\theta} \\ \gamma_{rx} \\ \gamma_{x\theta} \end{Bmatrix}^{(k)} \tag{A2}$$

where $c_{11}, c_{12}, \ldots, c_{66}$ are the material stiffness coefficients of the kth layer. The different stress resultants are obtained through piecewise integration in the thickness of eqns (A2).

Strain and kinetic energies

The total strain energy of the cylinder, U, can be decomposed into three components as follows:

$$U = U_{be} + U_{sh} + U_{tn} \tag{A3}$$

where U_{be}, U_{sh}, and U_{tn} are the bending-extensional, transverse shear, and transverse normal energies, respectively. The expressions of U_{be}, U_{sh}, and U_{tn} in terms of the stress and strain components are given by:

$$\begin{Bmatrix} U_{be} \\ U_{sh} \\ U_{tn} \end{Bmatrix} = \frac{1}{2}\int_0^L \int_0^{2\pi} \sum_{k=1}^{NL} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + \sigma_{x\theta}\gamma_{x\theta} \\ \sigma_{rx}\gamma_{rx} + \sigma_{r\theta}\gamma_{r\theta} \\ \sigma_r\varepsilon_r \end{Bmatrix}^{(k)} r_0\left(1 + \frac{x_3}{r_0}\right) \mathrm{d}x_3\, \mathrm{d}\theta\, \mathrm{d}x \tag{A4}$$

where NL is the total number of layers.

The expression for the total kinetic energy of the cylinder, K, is given by:

$$K = \frac{1}{2}\int_0^L \int_0^{2\pi} \sum_{k=1}^{NL} \int_{h_{k-1}}^{h_k} \rho^{(k)}[(\partial_t u)^2 + (\partial_t v)^2 + (\partial_t w)^2] r_0\left(1 + \frac{x_3}{r_0}\right) \mathrm{d}x_3\, \mathrm{d}\theta\, \mathrm{d}x \tag{A5}$$

where $\partial_t = \partial/\partial t$.

First-order shear-deformation theory

The first-order shear-deformation theory used herein is based on the following assumptions (Ambartsumian, 1974; Dong and Tso, 1972; Noor and Peters, 1989b):

(a) linear through-the-thickness variation of the displacements u and v;
(b) neglecting the transverse normal strain, ε_r; and
(c) generalized plane-stress state in each layer.

The displacement field is completely described by the five parameters $u^{(0)}$, $v^{(0)}$, $w^{(0)}$, $u^{(1)} = \phi_x^0$, $v^{(1)} = \phi_\theta^0$ where ϕ_x^0 and ϕ_θ^0 are average through-the-thickness rotations, and the fundamental equations of the theory are given in Dong and Tso (1972) and Noor and Peters (1989b).

Correction factors are used to adjust the transverse shear stiffnesses and match the response predicted by the two-dimensional theory with that of the three-dimensional elasticity theory. The range of validity of the first-order shear-deformation theory is strongly dependent on the factors used in adjusting the transverse shear stiffnesses of the cylinder.

Simplified higher-order theory

In the simplified higher-order theory used herein, assumption (a) of the first-order shear deformation theory is replaced by that of a cubic variation, through-the-thickness, for the in-plane displacements u and v. In order to retain the same number of displacement parameters as in the first-order theory, the transverse shear stress (and strain) conditions are imposed at the top and bottom surfaces of the cylinder (see, for example, Bhimaraddi, 1985; Khdeir *et al.*, 1989; Librescu *et al.*, 1989; Reddy and Liu, 1985).